A

NEW METHOD

OF RESOLVING

CUBIC EQUATIONS.

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A New Method *of resolving* Cubic Equations. *By James Ivory*, Esq. *Communicated by John Playfair*, F.R.S. Edin. *and Professor of Mathematics in the University of Edinburgh.*

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1. I Divide cubic equations into two varieties or species: the one, comprehending all cubic equations with three real roots; the other, all those with only one real root.

2. Let φ denote any angle whatever, and let $\tau = \tan\varphi$, the radius being unity: let also $z = \tan\frac{\varphi}{3}$: then from the doctrine of angular sections we have

$$\tau = \frac{3z - z^3}{1 - 3z^2},$$

which being reduced to the form of an equation, is

$$z^3 - 3\tau z^2 - 3z + \tau = 0.$$

Now, from what is commonly taught in angular sections, z, in this equation, may denote, not only $\tan\frac{\varphi}{3}$, but also $\tan\left(\frac{\varphi}{3} + 120°\right)$, or $\tan\left(\frac{\varphi}{3} + 240°\right)$. It is to be remarked, too, that any value whatsoever may be assigned to τ, positive or negative, and without limit or restriction as to magnitude. The equation, then, has three different values of z for every given value of τ; and it belongs to the species of cubic equations, having three real roots.

 3. Again

3. AGAIN I affume this expreffion,

$$\frac{1-\tau}{1+\tau} = \frac{(1-z)^3}{(1+z)^3}.$$

LET there be conceived an equilateral hyperbola, of which the femiaxes are each equal to unity, and let a ftraight line be drawn to touch the hyperbola at its vertex: Conceive alfo a ftraight line to be drawn from the centre, to cut off a fector from the hyperbola itfelf, or from its oppofite, or conjugate hyperbolas, and to intercept a part τ (eftimated from the vertex) on the tangent line: And, in like manner, let another ftraight line be drawn from the centre to cut off another fector, that fhall be one third part of the former fector, and to intercept a part z on the tangent line: Then the relation of τ and z will be as in the expreffion here affumed, viz.

$$\frac{1-\tau}{1+\tau} = \frac{(1-z)^3}{(1+z)^3}.$$

I SHALL not ftop to demonftrate this propofition refpecting the hyperbola: it eafily follows from the known properties of that curve. I mention it merely with the view of marking the ftrict analogy that fubfifts between the two varieties of cubic equations. It is fufficient for our purpofe to remark, what is indeed very evident from the nature of the affumed expreffion, that, whatever value be affigned to τ, z has always one real correfpondent value, and only one.

FROM our affumed expreffion we get

$$\tau = \frac{(1+z)^3-(1-z)^3}{(1+z)^3+(1-z)^3} = \frac{3z+z^3}{1+3z^2},$$

which being reduced to the form of an equation, is

$$z^3 - 3\tau z^2 + 3z - \tau = 0.$$

THIS equation has only one value of z for every given value of τ; and it belongs to the fpecies of cubic equations having only one real root.

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4. In order to give to the two equations, inveſtigated above, the utmoſt generality of which they are capable, I write $\frac{\tau}{R}$ for τ, and they finally become,

$$\text{I. } Rz^3 - 3\tau z^2 - 3Rz + \tau = 0,$$
$$\text{II. } Rz^3 - 3\tau z^2 + 3Rz - \tau = 0,$$

in which two equations, R and τ repreſent any numbers, poſitive or negative, and altogether unlimited and arbitrary as to magnitude.

I consider the two preceding equations as the ſimple caſes, or ſimple forms, of the two ſpecies of cubic equations: And the method of reſolution I have to propoſe is, to reduce every cubic equation whatſoever to one or other of theſe two forms.

The firſt of the above forms is an equation belonging to the circle. It expreſſes the relation between the tangent of an arch, and the tangent of the third part of that arch: and it has, in all caſes, three real roots. If we take the angle φ, of which the tangent is $\frac{\tau}{R}$, the radius being unity; the three roots of the equation, or the three values of z, are, tan $\frac{\varphi}{3}$, tan $\left(\frac{\varphi}{3} + 120^\circ\right)$, and tan $\left(\frac{\varphi}{3} + 240^\circ\right)$.

The ſecond of theſe forms is an equation belonging to the hyperbola. It expreſſes the relation between the tangents of two hyperbolic ſectors, of which the one is triple of the other; and it has, in all caſes, only one real root. From the expreſſion aſſumed (Art. 3.) above, whence this equation was deduced, we get, $\frac{1-z}{1+z} = \sqrt[3]{\frac{1-\tau}{1+\tau}}$: therefore, $z = \dfrac{1 - \sqrt[3]{\frac{1-\tau}{1+\tau}}}{1 + \sqrt[3]{\frac{1-\tau}{1+\tau}}}$: or, writing $\frac{\tau}{R}$ for τ, $z = \dfrac{1 - \sqrt[3]{\frac{R-\tau}{R+\tau}}}{1 + \sqrt[3]{\frac{R-\tau}{R+\tau}}}$: And ſo z is found by extracting the cubic root of a given number.

THE two forms differ from one another only in the ſigns of their terms. The firſt and third terms, as well as the ſecond and fourth, have always unlike ſigns in the firſt form; but always like ſigns in the ſecond form. This property reſpecting the ſigns of the alternate terms, by which the one equation is eſſentially diſtinguiſhed from the other, I ſhall denominate the "*Characteriſtic of the form.*"

5. I PROCEED, now, to ſhew, in what way any propoſed cubic equation may be reduced to one or other of the two forms.

LET the propoſed equation be,

$$x^3 + Ax^2 + Bx + C = 0,$$

where A, B, C denote any given coefficients, poſitive or negative. I aſſume $x = \frac{a+z}{b+z}$: a and b being indeterminate quantities, and z a new unknown quantity. And it is to be obſerved, that the ſuppoſition of $x = \frac{a+z}{b+z}$ is always poſſible, provided a be not equal to b: for if a be not equal to b, a value may be aſſigned to z, ſuch, that the fraction $\frac{a+z}{b+z}$ ſhall be equal to any number whatever, poſitive or negative. But if $a = b$, the value of $\frac{a+z}{b+z}$ is not altered, whatever number z may denote.

HAVING ſubſtituted, and taken away the denominators, we get,

$$(a+z)^3 + A(a+z)^2(b+z) + B(a+z)(b+z)^2 + C(b+z)^3 = 0.$$

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The terms of this expreſſion are now to be evolved and arranged, according to the powers of z: which being done, we ſhall find,

$$\left.\begin{array}{llll}
a^3 & +\ 3a^2\times z & +\ 3a\times z^2 & +\ 1\times z^3 \\
+\ Aa^2b & +\ 2Aab\times z & +\ Ab\times z^2 & \\
 & +\ Aa^2\times z & +\ 2Aa\times z^2 & +\ A\times z^3 \\
+\ Bab^2 & +\ Bb^2\times z & & \\
 & +\ 2Bab\times z & +\ 2Bb\times z^2 & \\
 & & +\ Ba\times z^2 & +\ B\times z^3 \\
+\ Cb^3 & +\ 3Cb^2\times z & +\ 3Cb\times z^2 & +\ C\times z^3
\end{array}\right\} = 0.$$

In order to reduce this equation to our forms, we muſt equate three times the coefficient of z^3 to the coefficient of z, either with the ſame or different ſigns: and alſo the coefficient of z^2 to three times the abſolute term, likewiſe with the ſame or different ſigns. For in the forms the coefficient of z^3 and z are R and $\mp$ 3R: and the coefficient of z^2 and the abſolute term are, $-\ 3r$ and $\pm\ r$. Now, in the transformed equation above, three times the coefficient of z^3 is $3 + 3A + 3B + 3C$, which I write thus, $(3 + 2A + B) + (A + 2B + 3C)$: And in like manner, for three times the abſolute term, I write $(3a^3 + 2Aa^2b + Bab^2) + (Aa^2b + 2Bab^2 + 3Cb^3)$. This being obſerved, we ſhall have theſe two equations for determining a and b:

$$\left.\begin{array}{r} 3 + 2A + B \\ + A + 2B + 3C \end{array}\right\} = \mp \left\{\begin{array}{l} 3a^2 + 2Aab + Bb^2 \\ Aa^2 + 2Bab + 3Cb^2, \end{array}\right.$$

$$\left.\begin{array}{r} 3a + 2Aa + Ba \\ + Ab + 2Bb + 3Cb \end{array}\right\} = \mp \left\{\begin{array}{l} 3a^3 + 2Aa^2b + Bab^2 \\ + Aa^2b + 2Bab^2 + 3Cb^3. \end{array}\right.$$

6. It is manifeſt, from the manner in which I have written the two equations for determining a and b, that they depend

upon

upon the two following more ſimple equations of the quadratic form :

$$3a^2 + 2Aab + Bb^2 = \mp (3 + 2A + B),$$
$$Aa^2 + 2Bab + 3Cb^2 = \mp (A + 2B + 3C).$$

For theſe two equations are no other than the two parts of the firſt of the preceding equations : and if we multiply the firſt of them by a, and the ſecond by b, we ſhall have the two parts of the ſecond of the preceding equations.

To determine a and b, I now write $M = 3 + 2A + B$; $N = A + 2B + 3C$; and $ay = b$: thus we have,

$$a^2 \times (3 + 2Ay + By^2) = \mp M,$$
$$a^2 \times (A + 2By + 3Cy^2) = \mp N.$$

Multiply the firſt equation by N ; the ſecond by M ; ſubtract the one from the other ; and divide by a^2 ; and there will reſult this equation for y :

$$(3N - AM) + 2 (AN - BM) y + (BN - 3CM) y^2 = 0.$$

And in this equation there is no ambiguity of ſigns.

If we ſuppoſe $y = 1$, the equation laſt found is equivalent to the identical equation $MN - MN = 0$. One value of y is therefore unity. But this is preciſely the caſe of $a = b$, which we have noticed above to be inapplicable to the preſent purpoſe. Nor is it to be wondered at that this value of y is of no uſe in the preſent inquiry : for it is manifeſt that it does not at all depend on the given quantities A, B, C, M and N, but merely on the peculiar manner in which the equation is conſtituted. We learn from hence, however, that there is always another value of y, whatever numbers A, B, C, M and N may denote ; becauſe every quadratic muſt have two roots or none at all.

Quadratic equations, with one root $= 1$, may be ſuppoſed to be thus generated: $(y - 1) \times (my - n) = my^2 - (m + n) y + n = 0$, the

the two roots being 1 and $\frac{n}{m}$: Comparing this formula with our equation, we have $n = 3N - AM$ and $m = BN - 3CM$: And ſo we get,

$$y = \frac{3N - AM}{BN - 3CM}.$$

Now, $a^2 = \frac{\mp M}{3 + 2Ay + By^2}$ and $b = a \times y$: therefore,

$$a = (BN - 3CM) \times \mp \sqrt{\frac{\mp M}{3(BN-3CM)^2 + 2A(3N-AM)(BN-3CM) + B(3N-AM)^2}}$$

$$b = (3N - AM) \times \pm \sqrt{\frac{\mp M}{3(BN-3CM)^2 + 2A(3N-AM)(BN-3CM) + B(3N-AM)^2}}$$

THE quantities a and b are therefore found by a ſingle extraction of the ſquare root: and they have each two values, one poſitive, and the other negative. It is indifferent which of theſe two values of a and b be taken, provided they are correſpondent values, ſo that $b = a \times y$. It is to be remarked too, that a and b have always real values, on account of the double ſign prefixed to M; for that ſign is to be taken that will render the radical quantity poſitive. And it is to be carefully noted which of the two ſigns is neceſſary, that a and b may have real values: becauſe on this depends the characteriſtic of the reduced equation, and whether it is to be referred to the firſt or ſecond form, and, conſequently, whether it has three roots, or only one. If the ſign $-$ is requiſite that a and b may have real values, then the reduced equation will have the characteriſtic of the firſt form, and will have three roots. But if the ſign $+$ is requiſite for that end, the reduced equation will have the characteriſtic of the ſecond form, and will have only one root. All this is manifeſt from the ſtatement in Art. 5.

7. THE rule, or law, according to which the preceding formulæ for a and b are conſtituted, is ſufficiently ſimple and perſpicuous; and the formulæ are therefore, in that reſpect, convenient for practice. But in examining the expreſſion in the denominator

denominator of the radical, I find that it is always divifible by M, the quantity in the numerator: and that thus the formulæ may be exhibited in another fhape, having this advantage, that it will introduce fmaller numbers in the arithmetical operations requifite for computing a and b.

I WRITE $Q \times M = 3(BN - 3CM)^2 + 2A(BN - 3CM)(3N - AM) + B(3N - AM)^2$, and evolving by actual multiplication

$$\begin{aligned} Q \times M = \; & 3B^2 . N^2 - 18BC . MN + 27C^2 . M^2 \\ & + 6AB . N^2 - 2A^2B . MN \\ & \qquad\qquad - 18AC . MN + 6A^2C . M^2 \\ & + 9B . N^2 - 6AB . MN + BA^2 . M^2 \end{aligned}$$

Now, the coefficient of N^2 is, $3B^2 + 6AB + 9B = 3B(B + 2A + 3) = 3B \times M$: dividing therefore by M, we get,

$$\begin{aligned} Q = 3B \times N^2 & - (18BC + 2A^2B + 18AC + 6AB) \times N \\ & + (27C^2 + 6A^2C + BA^2) \times M. \end{aligned}$$

And if for N and M we fubftitute their values $(A + 2B + 3C)$ and $(3 + 2A + B)$, we fhall find,

$$Q = 12B^3 + 81C^2 - 54ABC + 12A^3C - 3A^2B^2$$

all the other terms deftroying one another, except thefe five.

ALL the terms in this value of Q, being divifible by 3, I change Q, and put

$$Q = 4B^3 + 27C^2 - 18ABC + 4A^3C - A^2B^2$$

$$\text{or, } Q = (4B^3 + 27C^2) - 2AC(9B - A^2) + A^2(2AC - B^2)$$

And fince 3QM is now equal to the denominator of the radical in the preceding formulæ for a and b, we have the following new formulæ,

$$a = \pm \frac{BN - 3CM}{\sqrt{\mp 3Q}}.$$

$$b = \pm \frac{3N - AM}{\sqrt{\mp 3Q}}.$$

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WHAT was remarked above, with regard to the double ſign prefixed to M, is now to be applied to the double ſign prefixed to Q.

8. THE preceding inveſtigation ſupplies us with the following rule, or criterion, by which to determine, whether any propoſed cubic equation has three real roots or not:

THE propoſed equation will have three real roots, when the amount of the expreſſion

$$(4B^3 + 27C^2) - 2AC(9B - A^2) + A^2(2AC - B^2)$$

is negative: But if this expreſſion is poſitive, the equation will have only one real root: And, (as will afterwards be ſhewn), when the amount of the expreſſion is $= 0$, the equation will have two equal roots.

9. HAVING now found a and b, if we ſubſtitute $\frac{a+z}{b+z}$ for x in the propoſed equation, we ſhall have an equation for z that will come under one of our two forms, and from which z (and conſequently the root or roots of the propoſed equation) may therefore be found. But ſuch ſubſtitution is not neceſſary. For if we go back to Art. 5. and compare the transformed equation with the forms, we ſhall find $R = 1 + A + B + C$, and $3R = M + N$: alſo $-3r = (3a + 2Aa + Ba)$ $+ (Ab + 2Bb + 3Cb) = Ma + Nb$: Therefore

$$\frac{r}{R} = -\frac{Ma + Nb}{M + N}.$$

Whence the value or values of z are found by what is obſerved in Art. 4.

10. I SHALL now give the reſult of the whole analyſis in the form of a general rule for the reſolution of cubic equations, and add a few examples by way of illuſtration.

LET the propoſed equation be

$$x^3 + Ax^2 + Bx + C = 0.$$

1. COM-

1. COMPUTE, $M = 3 + 2A + B$

$$N = A + 2B + 3C$$

$$m = BN - 3CM$$

$$n = 3N - AM,$$

And $Q = (4B^3 + 27C^2) - 2AC(9B - A^2) + A^2(2AC - B^2)$

AND let it be carefully noted under which of the two following cafes the equation comes:

CASE I. When Q is negative.

CASE II. When Q is pofitive.

To thefe two cafes a third may be added, viz. when $Q = 0$: but of this cafe I fhall treat in one of the following examples.

2. COMPUTE alfo, $a = \frac{\pm m}{\sqrt{\mp 3Q}}$

$$b = \frac{\pm n}{\sqrt{\mp 3Q}} = \frac{an}{m}$$

and $\tau = -\frac{Ma + Nb}{M + N}$.

3. THEN, if the equation comes under cafe I: Find the angle φ, of which the tangent is τ, the radius being unity: take $z = \tan\frac{\varphi}{3}$, $z = \tan\left(\frac{\varphi}{3} + 120^\circ\right)$, and $z = \tan\left(\frac{\varphi}{3} + 240^\circ\right)$: And the three roots of the equation will be found, by fubftituting thefe values of z in the formula $x = \frac{a+z}{b+z}$.

4. BUT if the equation comes under Cafe II. we muft compute

$$z = \frac{1 - \sqrt[3]{\frac{1-\tau}{1+\tau}}}{1 + \sqrt[3]{\frac{1-\tau}{1+\tau}}},$$

And the only root of the equation will be found, by fubftituting this value of z in the formula $x = \frac{a+z}{b+z}$.

THE

THE following examples are chiefly taken from Dr HUTTON's "*Tracts Mathematical and Philosophical.*" *Tract* 5.

11. EXAMPLE 1. Let the proposed equation be

$$x^3 - 6x^2 + 11x - 6 = 0.$$

Here, A = — 6, B = + 11, C = — 6: therefore

$$M = +2$$
$$N = -2$$
$$m = +14$$
$$n = +6$$
$$Q = -4$$

So that we have here Case I.

$$a = \frac{14}{2\sqrt{3}} = \frac{7}{\sqrt{3}}$$
$$b = \frac{6}{14} \times a = \sqrt{3}$$

$$\text{and } \tau = -\frac{2.\frac{7}{\sqrt{3}} - 2.\sqrt{3}}{2-2} = \infty$$

Now, tan 90° = ∞, and also tan 270° = ∞: And we may take either of these angles for φ: Take $\varphi = 90°$, then,

$$z = \tan\frac{\varphi}{3} = \tan 30° = +\frac{1}{\sqrt{3}}$$
$$z = \tan\left(\frac{\varphi}{3} + 120°\right) = \tan 150° = -\tan 30° = -\frac{1}{\sqrt{3}}$$
$$z = \tan\left(\frac{\varphi}{3} + 240°\right) = \tan 270° = \infty$$

$$\text{THEREFORE, } x = \frac{\frac{7}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}} = \frac{8}{4} = 2$$

$$x = \frac{\frac{7}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{\sqrt{3} - \frac{1}{\sqrt{3}}} = \frac{6}{2} = 3$$

$$x = \frac{\frac{7}{\sqrt{3}} + \infty}{\sqrt{3} + \infty} = 1.$$

And so the three roots are 1, 2, and 3.

SINCE $M+N=3(1+A+B+C)$: it is obvious, that when $M+N=0$, we ſhall alſo have $1+A+B+C=0$; and ſo one root of the equation is unity. It is manifeſt too, that, in this caſe, the transformed equation for z becomes ſimply, $3z^2-1=0$: whence $z=\pm\frac{1}{\sqrt{3}}$; and conſequently the other two roots of the equation are, $x=\dfrac{a+\frac{1}{\sqrt{3}}}{b+\frac{1}{\sqrt{3}}}$, and $x=\dfrac{a-\frac{1}{\sqrt{3}}}{b-\frac{1}{\sqrt{3}}}$.

Or, writing for a and b, their values, $\frac{m}{\sqrt{-3Q}}$ and $\frac{n}{\sqrt{-3Q}}$, the two roots are, $x=\frac{m+\sqrt{-Q}}{n+\sqrt{-Q}}$ and $x=\frac{m-\sqrt{-Q}}{n-\sqrt{-Q}}$.

12. EXAMPLE 2. Let there be propoſed

$$x^3-6x^2+9x-2=0.$$

Here $A=-6$; $B=+9$; $C=-2$: And hence,

$$M=0$$
$$N=+6$$
$$m=+54$$
$$n=+18$$
$$Q=-108.$$

And we have here again Caſe I.

$$a=+3$$
$$b=+1$$
$$\tau=-1$$

Therefore $\varphi=$ Arc. tan $-1=145°$, therefore,

$$z=\tan\frac{\varphi}{3}=\tan 45°=1,$$

$$z=\text{t.}\left(\frac{\varphi}{3}+120°\right)=\text{t.}\ 165°=-\text{t.}\ 15°=-(2-\sqrt{3}),$$

$$z=\text{t.}\left(\frac{\varphi}{3}+240°\right)=\text{t.}\ 285°=-\text{t.}\ 75°=-(2+\sqrt{3});$$

THEREFORE, $x=\frac{3+1}{1+1}=\frac{4}{2}=2$;

$$x = \frac{3 - 2 + \sqrt{3}}{1 - 2 + \sqrt{3}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)^2}{2} = 2 + \sqrt{3};$$

$$x = \frac{3 - 2 - \sqrt{3}}{1 - 2 - \sqrt{3}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{2} = 2 - \sqrt{3};$$

And the three roots are 2, $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

13. I TAKE the following example to illuſtrate the caſe of two equal roots.

EXAMPLE 3. Let the equation be

$$x^3 - 7x^2 - 5x + 75 = 0.$$

Here $A = -7$: $B = -5$: $C = +75$. Therefore,

$$M = -16$$
$$N = +208$$
$$m = +2560$$
$$n = +512$$
$$Q = 0.$$

SINCE, then, $Q = 0$, it is manifeſt that a, b, and τ will be all infinitely great. But though a and b be infinite, it is to be remarked that $\frac{a}{b} = \frac{m}{n}$.

SINCE $\tau = \infty$, we have $\varphi = 90°$, or $\varphi = 270°$: Take $\varphi = 90°$: Then,

$$z = \tan\frac{\varphi}{3} = \tan 30 = \frac{1}{\sqrt{3}}$$

$$z = \tan\left(\frac{\varphi}{3} + 120°\right) = \tan 150° = -\tan 30° = -\frac{1}{\sqrt{3}}$$

$$z = \tan\left(\frac{\varphi}{3} + 240°\right) = \tan 270° = \infty$$

WE have then, in the firſt place, for two roots,

$$x = \frac{a + \frac{1}{\sqrt{3}}}{b + \frac{1}{\sqrt{3}}}$$

$$x = \frac{a - \frac{1}{\sqrt{3}}}{b - \frac{1}{\sqrt{3}}}.$$

And

And ſince a and b are infinitely great, it is manifeſt that theſe two roots are equal to one another, and each $= \frac{a}{b} = \frac{m}{n}$.

We can infer nothing with regard to the value of the third root, derived from the infinite tangent, unleſs we can aſcertain the proportion which that infinite tangent bears to the infinitely great quantities a and b. The general relation of $\tau = \tan\varphi$, and $z = \tan\frac{\varphi}{3}$ is thus expreſſed $\tau = \frac{3z - z^3}{1 - 3z^2}$: and τ becomes infinitely great; 1ſt, When $1 - 3z^2 = 0$; 2dly, When z is infinitely great.

Now the values of z, derived from the equation $1 - 3z^2 = 0$, are $z = + \frac{1}{\sqrt{3}}$, and $z = - \frac{1}{\sqrt{3}}$: And theſe are preciſely the values of z uſed above in determining the two equal roots.

Again, we have $\frac{\tau}{z} = \frac{3 - z^2}{1 - 3z^2}$: And it is manifeſt that the greater z is, the nearer $\frac{\tau}{z}$ approaches to $\frac{1}{3}$: ſo that, ultimately, when τ and z are greater than any finite magnitudes, we have $\frac{\tau}{z} = \frac{1}{3}$ and $z = 3\tau$. But $\tau = \frac{-Ma - Nb}{M + N}$; therefore, $z = \frac{-3Ma - 3Nb}{M + N}$ and $x = \frac{a + z}{b + z} = \frac{a - \frac{3Ma + 3Nb}{M + N}}{b - \frac{3Ma + 3Nb}{M + N}}$. If now we write $\frac{an}{m}$ for b, in this expreſſion, the infinite quantity a may be thrown out by diviſion, and we ſhall have the value of the root in finite quantities only. The expreſſion being properly reduced, we ſhall have, $x = \frac{2Mm + N(3n - m)}{M(3m - n) + 2Nn}$.

When, therefore, $Q = 0$, two roots are equal to one another, and each $= \frac{m}{n}$: And we have this formula by which to compute the third root, $x = \frac{2Mm + N(3n - m)}{2Nn + M(3m - n)}$.

Applying

APPLYING this rule to our example, we have:

for the two equal roots, $x = \frac{m}{n} = \frac{2560}{512} = 5$

for the third root, $x = \frac{-81920 - 212992}{-114688 + 212992} = \frac{-294912}{+98304} = -3$

14. EXAMPLE 4. Let the equation be

$$x^3 - 7x^2 + 18x - 18 = 0.$$

Here A = − 7; B = + 18; C = − 18. Therefore,

$$M = +7$$
$$N = -25$$
$$m = -72$$
$$n = -26$$
$$Q = +72.$$

So that we have here Case II.

$$a = \frac{-72}{\sqrt{216}} = \frac{-36}{3\sqrt{6}}$$

$$b = \frac{-26}{\sqrt{216}} = \frac{-13}{3\sqrt{6}}$$

$$\tau = +\frac{73}{54\sqrt{6}}$$

Therefore, $\sqrt[3]{\frac{1-\tau}{1+\tau}} = \sqrt[3]{\frac{54\sqrt{6}-73}{54\sqrt{6}+73}} = \frac{2\sqrt{6}-1}{2\sqrt{6}+1}$

HENCE $z = \frac{1-\sqrt[3]{\frac{1-\tau}{1+\tau}}}{1+\sqrt[3]{\frac{1-\tau}{1+\tau}}} = \frac{1}{2\sqrt{6}}$

and $x = \frac{-\frac{36}{3\sqrt{6}}+\frac{1}{2\sqrt{6}}}{-\frac{13}{2\sqrt{6}}+\frac{1}{2\sqrt{6}}} = \frac{-72+3}{-26+3} = \frac{-69}{-23} = +3.$

And 3 is the only root of the equation.

15. WHEN τ is a surd as $\frac{p}{q\sqrt{r}}$, the value of z (in Case II.) always involves radicals of this form, $\sqrt[3]{\frac{q\sqrt{r}-p}{q\sqrt{r}+p}}$; out of which the root may sometimes be extracted; and so the value of z will be expressed by a surd of the same kind as τ.

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The method I have followed to find when this can be done, being very ſimple and eaſy in practice, I ſhall here briefly deſcribe it.

We have $\tau = \frac{3z + z^3}{1 + 3z^2}$;

write $\frac{p}{q\sqrt{r}}$, for τ, $\frac{\mu}{\nu\sqrt{r}}$ for z : then

$$\frac{p}{q\sqrt{r}} = \frac{3r\mu\nu + \mu^3}{(r\nu^3 + 3\mu^2\nu)}\sqrt{r},$$

whence theſe two equations are formed.

$$p = 3r\mu\nu + \mu^3$$
$$q = r\nu^3 + 3\mu^2\nu,$$

from which it is manifeſt, that μ is a diviſor of p, and ν a diviſor of q. I ſeek then amongſt the diviſors of p for a number μ, and amongſt the diviſors of q for a number ν, that will ſatisfy the two equations above: or rather, that will ſatisfy theſe two following,

$$\nu^2 = \frac{p - \mu^3}{3r\mu},$$

$$\mu^2 = \frac{q - r^3}{3\nu}.$$

If two ſuch numbers are to be found amongſt the diviſors of p and q, then will $z = \frac{\mu}{\nu\sqrt{r}}$: but if not, we are to conclude that the value of z cannot be expreſſed this way.

Thus, in the laſt example $\tau = \frac{73}{54\sqrt{6}}$, it is manifeſt that 73 admits no diviſor but 1 : therefore $\mu = 1$, and on trial I find $\nu = 2$, which two numbers ſatisfy the two equations, and therefore $z = \frac{1}{2\sqrt{6}}$.

16. The ſame method applies alſo to Caſe I. For, in this caſe,

$$\tau = \frac{3z - z^3}{1 - 3z^2}$$

and

and ſubſtituting $\frac{p}{q\sqrt{r}}$ for τ, and $\frac{\mu}{\nu\sqrt{r}}$ for z, we derive theſe two equations,

$$p = 3r\mu\nu^2 - \mu^3$$
$$q = r\nu^3 - 3\mu^2\nu,$$

whence it is manifeſt that μ is a diviſor of p, and ν a diviſor of q, as before. It will be eaſier for trial to write the equations thus:

$$\nu^2 = \frac{p + \mu^3}{3r\mu}.$$
$$\mu^2 = \frac{r\nu^3 - q}{3\nu}$$

AND let it be obſerved, that we may here give to μ and ν any ſigns conſiſtent with the condition, that $\frac{p+\mu^3}{3r\mu}$ and $\frac{r\nu^3 - q}{3\nu}$ (the values of ν^2 and μ^2) are poſitive numbers.

IT is to be remarked too, that, in this caſe, z has three values. If, however, we can find one value this way, the two others are readily obtained. For if v be one value of z, the two other values are $\frac{v + \sqrt{3}}{1 - v\sqrt{3}}$ and $\frac{v - \sqrt{3}}{1 + v\sqrt{3}}$: becauſe theſe values are the tangents of two arches that differ from the arch of which v is the tangent by 120° and 240°.

THOUGH this is a matter more curious than uſeful, I ſhall add one more example for the ſake of illuſtration.

LET the equation be

$$x^3 - 39x^2 + 479x - 1881 = 0.$$

Here A = − 39; B = + 479; C = − 1881: And

$$M = +\ 404$$
$$N = -\ 4724$$
$$m = +\ 16976$$
$$n = +\ 1584$$
$$Q = -\ 25600.$$

And the equation belongs to Caſe I.

$$a = \frac{16976}{160\sqrt{3}} = \frac{1061}{10\sqrt{3}}$$

$$b = \frac{1584}{160\sqrt{3}} = \frac{99}{10\sqrt{3}}$$

$$\tau = \frac{-39032}{-43200\sqrt{3}} = -\frac{4879}{5400\sqrt{3}}$$

To find whether z can be expreſſed by a ſurd as $\frac{\mu}{\nu\sqrt{r}}$: I write $p = -4879$, $q = 5400$, and $r = 3$; and I have theſe two equations,

$$\nu^2 = \frac{-4879 + \mu^3}{9\mu}$$

$$\mu^2 = \frac{3\nu^3 - 5400}{3\nu}$$

I find, that 7, 17, and 41, are diviſors of 4879. It is evident that $+7$ would give ν^2 negative: I therefore try -7, and it does not ſucceed: Neither does $+17$; but trying -17, I find $\nu^2 = 64$, and conſequently $\nu = \pm 8$: And -8 is found to ſucceed in the other equation: therefore $\mu = -17$ $\nu = -8$, and $z = \frac{17}{8\sqrt{3}}$.

We have then, $z = \frac{17}{8\sqrt{3}} = v$

$$z = \frac{v - \sqrt{3}}{1 + v\sqrt{3}} = \frac{-7}{25\sqrt{3}}$$

$$z = \frac{v + \sqrt{3}}{1 - v\sqrt{3}} = \frac{41}{-9\sqrt{3}}$$

Therefore, $x = \dfrac{\frac{1061}{10\sqrt{3}} + \frac{17}{8\sqrt{3}}}{\frac{99}{10\sqrt{3}} + \frac{17}{3\sqrt{3}}} = \dfrac{8488 + 170}{792 + 170} = \dfrac{8658}{962} = 9$

$$x = \dfrac{\frac{1061}{10\sqrt{3}} - \frac{7}{25\sqrt{3}}}{\frac{99}{10\sqrt{3}} - \frac{7}{25\sqrt{3}}} = \dfrac{5305 - 14}{495 - 14} = \dfrac{5291}{481} = 11$$

$$x = \dfrac{\frac{1061}{10\sqrt{3}} - \frac{41}{9\sqrt{3}}}{\frac{99}{10\sqrt{3}} - \frac{41}{9\sqrt{3}}} = \dfrac{9549 - 410}{891 - 410} = \dfrac{9139}{481} = 19.$$

And ſo the three roots are, 9, 11, and 19.

FINIS.